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A note on the westward drift of the earth's magnetic field

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A model of the earth's liquid core is assumed in which the underlying magnetic field and velocity are zonal and axially symmetric. Alfvén waves that vary as $e^{i(k\phi-\sigma t)}$ are considered, where ϕ is the angle of longitude. Buoyancy and Coriolis forces $\mathbf{\Omega} \times \mathbf{U}$ are included.

For a wide class of basic states and regions of flow, it is shown that roughly as many of the waves with a given $k \geqslant 2$ propagate eastwards as propagate westwards. All these waves are neutrally stable. The class of basic states is restricted by certain inequalities involving their velocity, magnetic field and entropy gradient.

It is observed that the known equivalence (Malkus 1967) between Alfvén waves with frequencies $\sigma \ll \Omega$ and inertial waves with frequencies σ_p which are $O(\Omega)$ still holds when buoyancy forces are present. The equivalence requires σ_p^2 to be real. If σ_p is pure imaginary, as is possible (though perhaps uncommon) in an unstably stratified medium, then the corresponding Alfvén wave is not neutrally stable and travels westwards.

1. Introduction

The gross features of the magnetic field at the earth's surface drift slowly westward. The speeds of the various components of a representation of the magnetic potential in spherical harmonics differ. In the decade 1955–65, the equatorial dipole had a speed of about 0·06° longitude per year, other of the lower-order components being faster by a factor of up to about five (Hide 1966). The axial dipole also varies but, except possibly during its sporadic reversals of direction, does so on a much longer scale of time.

Following the belief that the earth's magnetic field originates in an inner core composed mainly of iron, Braginskiĭ (1967), Hide (1966) and others have suggested that the drift might represent hydromagnetic waves superimposed on the mean (relatively steady) magnetic field. To propagate Alfvén waves with phase speeds of the order of magnitude of the observed drift rates, the zonal field needs a strength of about 100 Oersted. For a field of this strength, the Coriolis forces are of the same order of magnitude as the changes in the Lorentz forces due to the waves. Thus the earth's rotation is brought into the dynamics, and east and west are dynamically distinguished. A zonal field $\mathbf{B}_0$ of 100 Oersted would be roughly 20 times as big as the magnetic field $\mathbf{B}_p$ in the mantle at the core boundary. However this ratio $|\mathbf{B}_0|/|\mathbf{B}_p|$ can be regarded (e.g. Hide 1966) as a measure of the magnetic Reynolds number R_m of the geodynamo, and a figure of 20 for R_m would not be unacceptable (Gubbins 1974; Acheson & Hide 1973). Predictions of the direction of (zonal) propagation of the Alfvén waves vary, depending on the model proposed.

The simpler models omit buoyancy and acceleration relative to the earth and presuppose a basic state in which the fluid is stationary and the magnetic field is zonal and axially symmetric, i.e. $\mathbf{B}_0 = B_0(r,z)\,\hat{\boldsymbol{\phi}}$, where (r,ϕ,z) are cylindrical co-ordinates with origin at the earth's centre and $\hat{\mathbf{z}}$ directed northwards. Waves are sought that vary as $e^{i(k\phi-\sigma t)}$. For $B_0 = \text{constant} \times r$, Malkus found (shelloidal) waves that have no radial components of velocity or magnetic field. These modes can occur in a sphere or spherical annulus (and with spherical stratification) and they all travel eastwards. For a thin spherical annulus, all the waves determined from the appropriate approximate equations also travel eastwards, both for $B_0 \propto r$ and $B_0 = \text{constant}$ (Stewartson 1967). Stewartson argued that further, when $k > 2$ and $B_0 = \text{constant}$, each of these modes retains its easterly direction of propagation when the annulus widens and becomes relatively thick. These modes are special, in that they either have zero or small radial velocities or are generated from such modes. For $B_0 \propto r$, Malkus (1967) concluded that, apart from the shelloidal modes, the modes in a sphere exhibit no preferred direction of drift, one class of modes drifting to the east and another class drifting to the west. All the modes alluded to so far are neutrally stable ($\text{Im}\,\sigma = 0$). Acheson considered waves in a cylindrical annulus $r_1 < r < r_2$ that vary as $e^{i(k\phi+lz-\sigma t)}$ and showed that, for an axially symmetric basic state, the *unstable* waves propagate westwards (Acheson 1972, 1973). The basic state may be (cylindrically) stratified and include an axial and a zonal component of velocity. Whether or not a similar conclusion holds for unstable waves in other geometries is an unresolved question (Acheson & Hide 1973).

In the present note, Malkus's conclusion concerning the directions of drift in a sphere is generalized. The basic state is again assumed to be axially symmetric and to have a zonal magnetic field $\mathbf{B}_0 = B_0\hat{\boldsymbol{\phi}}$, but B_0 is a general function of r and z and a zonal velocity and buoyancy forces are included. The boundary of the fluid is again required to be rigid and axisymmetric. Otherwise its shape is arbitrary. The basic state is restricted by certain inequalities involving its velocity, magnetic field and entropy gradient. These inequalities involve the wavenumber k. For a *given* k, a basic state that satisfies these inequalities will be referred to as an *admissible state*. For admissible states, it is found that almost as many waves, with a given wavenumber, travel eastwards as travel westwards. All the waves in the admissible configurations are neutrally stable.

In passing, an extension is noted of the known equivalence (Malkus 1967) between Alfvén waves in a rotating magnetic fluid with $\mathbf{B}_0 \propto r\hat{\boldsymbol{\phi}}$ and inertial waves in a rotating non-magnetic fluid. This equivalence requires the phase speed of the Alfvén waves to be much less than the angular velocity $\boldsymbol{\Omega}$ of the fluid, so that their inertial acceleration can be neglected. The low frequency of the Alfvén wave then corresponds to a frequency $O(\Omega)$ of the inertial wave. It is observed that this equivalence still holds when stratification is included. A possible direct connexion is thereby opened up between the atmosphere and the liquid core.

2. Formulation

The relevant magnetohydrodynamic equations may be written as

$$\frac{d\mathbf{U}}{dt}+2\mathbf{\Omega}\times\mathbf{U} = -\frac{1}{\rho}\nabla p+\frac{1}{\rho}\,\mathrm{curl}\left(\frac{\mathbf{B}}{\mu}\right)\times\mathbf{B}-\nabla\Phi, \tag{2.1}$$

$$d\mathbf{B}/dt = \mathbf{B}.\nabla\mathbf{U}, \quad dS/dt = 0, \tag{2.2),(2.3}$$

$$d\rho/dt+\rho\,\mathrm{div}\,\mathbf{U} = 0 = \mathrm{div}\,\mathbf{B}, \tag{2.4}$$

where p, ρ, S, $\mathbf{U}$, $\mathbf{B}$, μ and Φ denote respectively pressure, density, entropy per unit mass, velocity relative to the earth, magnetic induction, magnetic permeability and the potential due to both gravity and centripetal acceleration; $\mathbf{g} = \nabla\Phi$ is taken positive outwards from the earth's centre, and the earth's angular velocity $\mathbf{\Omega}$ is regarded as constant. Buoyancy will be ascribed solely to entropy gradients, but the ensuing analysis still applies, with little more than a change of notation, if buoyancy is caused also by concentration gradients.

The Coriolis acceleration must be retained if east and west are to be dynamically distinguished. So the time scale for changes in (absolute) velocity in (2.1) is effectively 1 day, in contrast to the time scale t^* of 10^2–10^3 years characteristic of the present rate of westward drift. The time scale for changes in $\mathbf{B}$ and S will be taken to be t^*. The magnetic Reynolds number and Péclet number for the liquid core, based on a time of 10^2 years, an outer radius $(r^2+z^2)^{\frac{1}{2}} = a$ of 3×10^8 cm, a magnetic diffusivity of $3\times10^4\,\mathrm{cm^2\,s^{-1}}$ (Roberts & Soward 1972) and a thermal diffusivity of $10^{-2}\,\mathrm{cm^2\,s^{-1}}$ (Stacey 1969), are 10^4 and 2×10^{10} respectively and are large enough to justify neglecting magnetic and thermal diffusion (other than in boundary layers or other such singular layers). A kinematic viscosity of about $10^{12}\,\mathrm{cm^2\,s^{-1}}$ would be needed to make the viscous forces comparable with the Coriolis force. Estimates of ν vary between wide limits, namely the value $10^{-3}\,\mathrm{cm^2\,s^{-1}}$ for molten iron at ordinary pressures and the value $10^9\,\mathrm{cm^2\,s^{-1}}$, above which compressional waves would attenuate more than is observed (cf. Hide 1971). However, the upper limit seems sufficiently far below 10^{12} to permit (a similar) neglect of viscous diffusion.

The basic state is taken to be steady, to be symmetric about the axis of rotation and to have

$$\mathbf{U}_0 = r\zeta_0(r,z)\,\hat{\boldsymbol{\phi}}, \quad \mathbf{B}_0 = rb_0(r,z)\,\hat{\boldsymbol{\phi}}, \tag{2.5}$$

where $\zeta_0 \ll \Omega$. The suffix zero will be taken to denote the value of a variable in this state. The actual velocity and magnetic field of the geodynamo are not known. Arguments suggesting that the magnetic field and velocity are approximately zonal have been advanced by Roberts & Soward (1972) and Hide (1966). The argument relating to $\mathbf{B}_0$ requires the magnetic Reynolds number of the geodynamo ($R_m = aU/\lambda$, where U is a speed typical of the dynamo) to be large. It has sometimes (Malkus 1967; Acheson 1972) been assumed that $\mathbf{U}_0 = 0$. In the context of the Alfvén-wave hypothesis for the westward drift of the earth's magnetic field, making this assumption means presupposing (*a*) that the observed drift speed represents a phase speed rather than a particle speed and (*b*) that the geodynamo is sustained with a magnetic Reynolds number much less than the values 10^2–10^3 associated with the drift speed. The simpler assumption that $\mathbf{U}_0 = 0$ thus requires $1 \ll R_m \ll 10^2$–10^3. The inclusion of a zonal flow

($\mathbf{U}_0 = r\zeta\hat{\boldsymbol{\phi}}$) imposes little extra complication and removes the need for the upper limit $R_m \leqslant 10^2$–10^3.

The equilibrium of the basic state requires that

$$\nabla\left(p_0 + \frac{r^2 b_0^2}{2\mu_0}\right) = -\rho_0 \nabla\Phi_0 - \frac{h_0 \mathbf{r}}{\mu_0} - \frac{r^2 b_0^2}{2}\nabla\frac{1}{\mu_0}, \tag{2.6}$$

where
$$h_0 = b_0^2 - 2\mu_0\rho_0\Omega\zeta_0. \tag{2.7}$$

The acceleration $\zeta_0^2\mathbf{r}$ is much less than $\Omega\zeta_0\mathbf{r}$ and has been omitted. From here on, we shall also neglect the variation in μ. The Lorentz forces $h_0\mathbf{r}/\mu_0$ in question are $O(\Omega\sigma a\rho_0)$, as will be seen presently [equation (2.19)], and for $t^* = 10^2 y$ are about $10^{-8}\rho_0 g$. So they cause only a minute departure from horizontal stratification. Accordingly, the pressure, density and entropy are to a close approximation functions $\bar{p}_0(\Phi_0), \bar{\rho}_0(\Phi_0)$ and $\bar{S}_0(\Phi_0)$, say, of the potential Φ_0. The small variation in $\rho_0 - \bar{\rho}_0$ around an equipotential surface is given by

$$\mathbf{g} \times \nabla(\rho_0 - \bar{\rho}_0) = \frac{r}{\mu_0}\frac{\partial h_0}{\partial z}\hat{\boldsymbol{\phi}}. \tag{2.8}$$

Small disturbances of the basic state are governed, to first order, by

$$2\rho_0\boldsymbol{\Omega} \times \mathbf{U}_1 = -\nabla\Pi_1 + \mu_0^{-1}(\mathbf{B}_0.\nabla\mathbf{B}_1 + \mathbf{B}_1.\nabla\mathbf{B}_0) + \alpha_0 T_1\mathbf{g}, \tag{2.9}$$

$$\partial\mathbf{B}_1/\partial t = \mathbf{B}_1.\nabla\mathbf{U}_0 - \mathbf{U}_0.\nabla\mathbf{B}_1 + \mathbf{B}_0.\nabla\mathbf{U}_1 - \mathbf{U}.\nabla\mathbf{B}_0, \tag{2.10}$$

$$\partial T_1/\partial t = -(T/c_p)_0\,\mathbf{U}_1.\nabla S_0 \quad (= -\mathbf{U}_1.\nabla\tau_0, \quad \text{say}), \tag{2.11}$$

$$\operatorname{div}\mathbf{U}_1 = 0 = \operatorname{div}\mathbf{B}_1, \tag{2.12}$$

where $\Pi_1 = p_1 + \rho_0\Phi_1 + \mu_0^{-1}\mathbf{B}_0\cdot\mathbf{B}_1$, T is the temperature, c_p is the specific heat per unit mass at constant pressure, $-\alpha_0/\rho_0$ is the coefficient of thermal expansion and the suffixes 1 connote the changes in value from those in the basic state. The Boussinesq approximation has been used here, i.e. the density has been taken to be uniform in the equations of motion (2.9) and (2.12) except in calculating the buoyancy force, and the respective changes ρ_1 and S_1 in density and entropy have been taken to be independent of compression. The terms omitted from each equation on this account are smaller than the terms retained at worst by the factor of about $\frac{1}{5}$ that represents the proportional change in density across the liquid core. The ratio $(T/c_p)_0$ in (2.11) is assumed to be uniform also (though allowing for its variation would involve little change in what follows). The acceleration $d\mathbf{U}_1/dt$, which has been omitted from (2.9), is of the order of $10^{-5}\boldsymbol{\Omega} \times \mathbf{U}$ for slow waves with periods $t^* \simeq 10^2$ years (and for $\zeta = O(1/t^*)$) and is insignificant. (The accelerations $\mathbf{r} \times d\boldsymbol{\Omega}/dt$ due to the 25800 year precession and to the Chandler wobble, which were omitted from (2.1), are of the order of 10^{-3} and 10^{-5} times $(a/t^*U_1)\,\boldsymbol{\Omega} \times \mathbf{U}$ respectively.) With the acceleration $d\mathbf{U}_1/dt$ absent, the equations of motion (2.9)–(2.12) are invariant under transformation to any frame with a constant angular velocity $\boldsymbol{\Omega}_1 \leqslant \boldsymbol{\Omega}$.

Since the Coriolis force $\rho_0\boldsymbol{\Omega} \times \mathbf{U}_1$ is exceedingly small, a minute non-uniformity in the entropy S_0 of the basic state suffices to produce buoyancy forces of comparable magnitude. Variations in τ_0 which are $O(a\Omega\rho_0/\alpha_0 t^* g)$ suffice and it will be assumed from here on that the variations are of this order. For the typical values $\alpha_0/\rho_0 = -4 \times 10^{-6}\,°\mathrm{C}^{-1}$ and $t^* = 10^2$ years, these variations amount to about $10^{-2}\,°\mathrm{C}$. The corresponding departures in the density ρ_0 from its adiabatic value are about $10^{-8}\rho_0$. The small change in entropy $S_0 - \bar{S}_0$ round an equipotential corresponding to the change $\rho_0 - \bar{\rho}_0$ may be determined, neglecting compressibility, from (2.8) ,whence

$$\mathbf{g} \times \nabla \tau_0 = \left(\frac{T}{c_p}\right)_0 \mathbf{g} \times \nabla (S_0 - \bar{S}_0) = -\left(\frac{1}{\alpha\mu}\right)_0 r \frac{\partial h_0}{\partial z} \hat{\boldsymbol{\phi}}. \quad (2.13)$$

Assuming, as we have, that the Coriolis, magnetic and buoyancy forces in (2.9) are of comparable magnitude, we may infer† that $\mathbf{g} \times \nabla \tau_0$ and $(a/\alpha\mu)_0 \, \partial h_0/\partial z \hat{\boldsymbol{\phi}}$ are also of comparable magnitude. The small deviation of Φ from spherical symmetry can therefore be ignored in (2.13) [and (2.9)] with little additional error, and from here on we put $\hat{\mathbf{g}} = \sin\theta \hat{\mathbf{r}} + \cos\theta \hat{\mathbf{z}}$, where θ is the colatitude.

After introducing the sinusoidally varying forms

$$(\mathbf{U}_1, \mathbf{B}_1, T_1, \Pi_1) = (\mathbf{u}_1, \mathbf{b}_1, t_1, \pi_1)\, e^{i(k\phi - \sigma t)}, \quad k \geqslant 0, \quad (2.14)$$

into the magnetic and thermal equations (2.10) and (2.11), we get

$$(k\zeta_0 - \sigma)\,\mathbf{b}_1 = kb\mathbf{u}_1 + ir\hat{\boldsymbol{\phi}}(\mathbf{u}_1 . \nabla b_0 - \mathbf{b}_1 . \nabla \zeta_0), \quad (k\zeta_0 - \sigma)\, t_1 = i\mathbf{u}_1 . \nabla \tau_0. \quad (2.15)$$

Following Frieman & Rotenberg (1960), we define a displacement $\boldsymbol{\eta}$ by

$$\boldsymbol{\eta} = (\sigma - k\zeta_0)^{-1}\mathbf{u}_1 + ir(\sigma - k\zeta_0)^{-2}\hat{\boldsymbol{\phi}}\mathbf{u}_1 . \nabla \zeta_0. \quad (2.16)$$

The displacement $\boldsymbol{\eta}$ satisfies $$\operatorname{div} \boldsymbol{\eta} = 0 \quad (2.17)$$

and leads to a useful simplification of (2.15). In terms of $\boldsymbol{\eta}$,

$$\mathbf{u}_1 = (\sigma - k\zeta_0)\boldsymbol{\eta} - ir\hat{\boldsymbol{\phi}}\boldsymbol{\eta} . \nabla \zeta_0, \quad (2.18a)$$

$$\mathbf{b}_1 = -kb_0\boldsymbol{\eta} - ir\hat{\boldsymbol{\phi}}\boldsymbol{\eta} . \nabla b_0, \quad (2.18b)$$

$$t_1 = -i\boldsymbol{\eta} . \nabla \tau_0. \quad (2.18c)$$

Substituting (2.14) into the dynamic equation (2.9), and then using the relations (2.18) for $\mathbf{u}_1$, $\mathbf{b}_1$ and t_1, we find that

$$k^2 b_0^2 \boldsymbol{\eta} - i(2\omega + k(b_0^2 + h_0))\,\hat{\mathbf{z}} \times \boldsymbol{\eta} = i\mu_0 \nabla^* \pi_1 + \mathbf{r}\boldsymbol{\eta} . \nabla h_0 - \beta \hat{\mathbf{g}}\boldsymbol{\eta} . \nabla \tau_0, \quad (2.19)$$

where $$\omega = \mu_0 \rho_0 \sigma \Omega, \quad \beta = \mu_0 \alpha_0 g, \quad \nabla^* \pi_1 = \pi_{1r}\hat{\mathbf{r}} + ikr^{-1}\pi_1 \hat{\boldsymbol{\phi}} + \pi_{1z}\hat{\mathbf{z}}. \quad (2.20)$$

We assume that $\mathbf{u} . \boldsymbol{\nu} = 0$ at the boundary S of the fluid, where $\boldsymbol{\nu}$ is normal to S, and deduce from (2.18a, b) that $$\boldsymbol{\eta} . \boldsymbol{\nu} = 0 = \mathbf{b}_1 . \boldsymbol{\nu} \quad \text{on} \quad S. \quad (2.21)$$

3. General observations

For b_0 = constant and $\zeta_0 = \tau_0 = 0$, Malkus (1967) found a direct equivalence between slow Alfvén waves with frequencies $\ll \Omega$ and inertial oscillations in a homogeneous non-magnetic fluid. We note here that this equivalence can be extended to include stratification.

For waves with frequencies $O(\Omega)$, the acceleration $d\mathbf{U}_1/dt$ should be restored to (2.9). The equation for $\mathbf{u}_1$ in planetary waves in a non-magnetic, rigidly rotating, stratified fluid follows on putting $b_0 = 0$ and $\zeta_0 = 0$. Thus

$$-\rho_0 \sigma^2 \mathbf{u}_1 - 2i\rho_0 \sigma \boldsymbol{\Omega} \times \mathbf{u}_1 = i\sigma \nabla^* \pi_1 - \alpha_0 \mathbf{g}\mathbf{u}_1 \cdot \nabla \tau_0. \quad (3.1)$$

† Independently of (2.13).

It is implicit here that the Boussinesq approximation is being applied also to the non-magnetic waves. That is, ρ_0 is being regarded as constant in the equations of motion for the non-magnetic waves save in the buoyancy term. Also the non-magnetic waves are being taken to have the same continuity equation ($\operatorname{div} \mathbf{u}_1 = 0$) and the same boundary condition ($\mathbf{u}_1 . \boldsymbol{\nu} = 0$ on S) as the Alfvén waves; and the ratio $(T/c_p)_0$ in (2.11) is again being taken to be uniform. Following Malkus, we compare these waves with Alfvén waves for a basic state in which $b_0 =$ constant and $\zeta_0 = 0$ (so that $h_0 = b_0^2$ and $\boldsymbol{\eta} = \sigma^{-1}\mathbf{u}_1$). For a given region V, (3.1) yields the same eigenmodes $\mathbf{u}_1$ for the non-magnetic waves as (2.19) yields for the Alfvén waves just specified provided that

$$(\Omega/\sigma)_p = -(\omega + kb_0^2)/k^2b_0^2, \tag{3.2a}$$

$$(\alpha_0 g/\rho_0 \sigma^2)_p \nabla\tau_{0p} = -(\mu_0\alpha_0 g/k^2b_0^2)\nabla\tau_0, \tag{3.2b}$$

where the suffixes p denote quantities pertaining to the non-magnetic waves. The Alfvén waves with frequencies

$$\sigma = -kb_0^2(k\Omega_p\sigma_p^{-1}+1)/\mu_0\rho_0\Omega \ll \Omega \tag{3.3}$$

are thus equivalent to the non-magnetic waves with frequencies $O(\Omega)$ in V.

If $\alpha_0 g/(\alpha_0 g)_p$ is taken to be constant, the proviso (3.2b) requires a linear relation between τ_0 and τ_{0p} but leaves the distribution of τ_0 arbitrary. If $\alpha_0 g/(\alpha_0 g)_p$ is taken to vary and to have a spherically symmetrical distribution, then (3.2b) requires the distributions of τ_0 and τ_{0p} to be spherically symmetric. The condition (3.2b) also requires σ_p^2 to be real and non-zero. When σ_p is real, σ is real and for the slower non-magnetic waves with $|\sigma_p| < k\Omega_p$, $\sigma_p\sigma$ is negative; i.e. both waves are neutrally stable and if one propagates eastwards the other propagates westwards. If σ_p ($\neq 0$) is pure imaginary, then $\operatorname{Im}\sigma \operatorname{Im}\sigma_p > 0$, i.e. both waves are unstable (or stable) and the Alfvén wave propagates westwards.

The Alfvén waves on a torus that Braginskiĭ (1967) has considered provide an example of Alfvén waves which have equivalent non-magnetic waves. Braginskiĭ takes the torus to have a rectangular cross-section ($r = s_0$, $s_0 + x_1$, $z = 0, z_1$, where s_0, x_1 and z_1 are constants) and to be thin ($x_1, z_1 \ll s_0$) and takes $\mathbf{g}$ and $\nabla\tau_0$ to be uniform and parallel to $\hat{\mathbf{z}}$. The corresponding non-magnetic waves have pure imaginary values of σ_p for large enough, destabilizing values of $\nabla\tau_{0p}$. So imaginary values of σ_p can occur. But they may be rare.

It is of some relevance to ask whether the general equations for $\boldsymbol{\eta}$ are hyperbolic. For this purpose, we write (2.19) in the form

$$A_1 u + iBv + Cw = i\mu_0\pi_{1r}, \tag{3.4a}$$

$$-iBu + A_2 v = -\mu_0\pi_1/r, \tag{3.4b}$$

$$Cu + A_3 w = i\mu_0\pi_{1z}, \tag{3.4c}$$

where (u, v, w) are the polar components of the displacement $\boldsymbol{\eta}$ and

$$\left.\begin{aligned} A_1 = k^2b_0^2 - rh_{0r} + \beta\tau_{0r}\sin\theta, \quad A_2 = k^2b_0^2, \quad A_3 = k^2b_0^2 + \beta\tau_{0z}\cos\theta, \\ B = 2\omega + k(b_0^2 + h_0), \quad C = \beta\tau_{0r}\cos\theta. \end{aligned}\right\} \tag{3.5}$$

The symmetric occurrence of C in (3.4) is a consequence of using (2.13). From (3.4), we find that

$$\left.\begin{aligned} u &= i\mu_0[A_2A_3\pi_{1r}+kA_3Br^{-1}\pi_1-A_2C\pi_{1z}]/\Delta, \\ v &= -\mu_0[A_3B\pi_{1r}+k(A_1A_3-C^2)\,r^{-1}\pi_1-BC\pi_{1z}]/\Delta, \\ w &= -i\mu_0[A_2C\pi_{1r}+kBCr^{-1}\pi_1+(B^2-A_1A_2)\,\pi_{1z}]/\Delta, \end{aligned}\right\} \tag{3.6}$$

where
$$\Delta = A_1A_2A_3-B^2A_3-C^2A_2. \tag{3.7}$$

Substituting these expressions into the reduced continuity equation (2.17) yields an equation for π_1 whose highest derivatives occur in the combination

$$A_2A_3\pi_{1rr}-2A_2C\pi_{1rz}+(A_1A_2-B^2)\,\pi_{1zz}.$$

When the waves are neutrally stable, all the coefficients of these derivatives are real and the equation for π_1 is locally hyperbolic wherever

$$b_0 \neq 0, \quad \Delta < 0. \tag{3.8}$$

General conditions under which this inequality is satisfied will be given below.

4. Frequency restrictions for $k \neq 0$

In order to obtain restrictions on the frequency ω for $k \neq 0$, we multiply (3.4a, b, c) by the respective components $(\bar{u}, \bar{v}, \bar{w})$ of the complex conjugate of the displacement $\boldsymbol{\eta}$, integrate over the volume V and use (2.17) and (2.21). In this way, we find

$$\int_V [A_1|u|^2+A_2|v|^2+A_3|w|^2+2B\,\mathrm{Im}\,u\bar{v}+2C\,\mathrm{Re}\,u\bar{w}]\,d\tau = 0. \tag{4.1}$$

The real part of the integrand is greater than or equal to

$$\frac{\Delta_R}{A_2A_3}|u|^2+A_2\left[|v|-\frac{|B_Ru|}{A_2}\right]^2+A_3\left[|w|-\frac{|Cu|}{A_3}\right]^2, \tag{4.2}$$

where
$$B_R = \omega_R+k(b_0^2+h_0), \quad \omega_R = \mathrm{Re}\,\omega, \tag{4.3}$$

and Δ_R is defined as in (3.7) but with B_R in place of B. Since $A_2\;(=k^2b_0^2)$ is non-negative, we conclude that if

$$b_0 \neq 0, \quad A_3 \geqslant 0 \tag{4.4}$$

everywhere in V (except possibly at a set of points whose aggregate volume is zero) then somewhere in V ($A_2 \neq 0, A_3 \neq 0$ and)

$$\Delta_R < 0. \tag{4.5}$$

If conditions (4.4) are met and, in addition,

$$(b_0^2+h_0)^2 < b_0^2(A_1-C^2A_3^{-1}) \quad (=D, \text{ say}) \tag{4.6}$$

everywhere in V, it follows from (3.5) and (4.5) that

$$2|\omega_R/k| > D^{\frac{1}{2}}-|b_0^2+h_0| > 0 \tag{4.7}$$

at the points in V where (4.5) holds. The first of these consequences of (4.1) [inequality (4.5)] implies that modes which are neutrally stable are governed by equations which are hyperbolic over part, at least, of V. The second [inequality (4.7)] places lower bounds on the zonal propagation speed $c_R = \mathrm{Re}\,\sigma/k$.

The basic states that comply with the prerequisites (4.4) and (4.6) are more easily recognized by restating these conditions in terms of the original variables b_0, ζ_0 and τ_0. From (2.13) and (3.5), we see that (4.4) requires that

$$b_0 \neq 0, \quad (A_3 =)\ k^2 b_0^2 + \beta\tau_{0z}\cos\theta \geqslant 0 \tag{4.8a, b}$$

and (4.6) requires that

$$\begin{aligned} k^2 b_0^2 &> b_0^{-2}(b_0^2+h_0)^2 + rh_{0r} - \beta\tau_{0r}\sin\theta + (\beta\tau_{0r}\cos\theta)^2 A_3^{-1} \\ &= b_0^{-2}(b_0^2+h_0)^2 + rh_{0r} - \beta\tau_{0r}\sin\theta(k^2b_0^2 + zh_{0z})\,A_3^{-1}. \end{aligned} \tag{4.9}$$

Condition (4.8b) is satisfied by all basic states that have a stabilizing axial entropy gradient ($z\tau_{0z} > 0$ for $z \neq 0$). Factors favourable to condition (4.9) being satisfied are as follows.

(i) A relatively small local difference between the zonal particle speed $a\zeta_0$ and the Alfvén speed $a(\mu\rho_0\Omega)^{-1}b_0^2$.

(ii) A large negative (or small positive) radial gradient of $(h_0 =)\,b_0^2 - 2\mu\rho_0\Omega\zeta_0$ relative to b_0^2/r.

(iii) A large positive (stabilizing) radial entropy gradient τ_{0r} relative to b_0^2/r for a given value of $E = (k^2b_0^2 + zh_{0z})/(k^2b_0^2 + \beta\tau_{0z}\cos\theta) > 0$, or a large negative (destabilizing) gradient τ_{0r} relative to b_0^2/r for a given value of $E < 0$.

The average magnitudes of $1 + h_0b_0^{-2}$, h_{0r}/b_0^2, h_{0z}/b_0^2 and τ_{0r}/b^2 can be varied independently. So the factors (i)–(iii) are essentially independent. If $|\mu\rho_0\Omega\zeta_0b_0^{-2} - 1| < 1$, $h_{0r} < 0$ and $\tau_{0r}E > 0$, then condition (4.9) is satisfied for all $k \geqslant 1$. (A horizontally stratified basic state ($\mathbf{g} \times \nabla\tau_0 = 0$) occurs in conjunction with a zero axial gradient of h_0, as is shown by (2.13). In this case, we have $E > 0$ whenever (4.8) applies.) Both of the conditions (4.8) and (4.9) are generally satisfied when $b_0 \neq 0$ anywhere in V and

(iv) k is large enough, or

(v) $k > 1$ and b_0 is large enough, for fixed h_0, or

(vi) $k > 2$ and b_0 is large enough, for fixed ζ_0.

The factors (ii)–(vi), which are strongly conducive to the condition (4.9) being satisfied, produce large values of $D/(b_0^2 + h_0)$, as may be seen by reverting to the antecedent (4.6) of (4.9), and hence produce large values of the lower bound to $|\omega_R/k(b_0^2 + h_0)|$ implied by (4.7). Thus these factors lead to Alfvén waves with high angular speeds c_R, relative to a typical magnitude of $|(\mu\rho_0\Omega)^{-1}b_0^2 - \zeta_0|$.

Directions can be assigned to the zonal velocities by appealing to a continuity argument that was used in a similar context by Stewartson (1967). We restrict attention to basic states that comply, for a given k, with conditions (4.8) and (4.9), and refer to them as admissible states. The inequality (4.7) shows that $\sigma_R \neq 0$. Hence the propagation speed c_R of a particular mode remains of one sign when the admissible basic states and the boundary of V change continuously, provided that the mode and its wave speed c_R also change continuously. Thus the mode propagates in the same zonal direction in the derived configuration as it does in the original configuration. Moreover, unless some of the modes fail to deform continuously, a complete set of modes with given k in one admissible configuration deforms to a complete set in any other admissible configuration.

A convenient prototype configuration is provided by the basic state

$$\zeta_0 = 0, \quad b_0 = \text{constant}, \quad \tau_0 = 0 \tag{4.10}$$

and the interior of a sphere. This state is admissible for $k > 2$, can be changed continuously to admissible states for $k = 2$, and is disjoint from admissible states for $k = 1$. Accordingly, only modes with $k \geqslant 2$ can be derived from this particular prototype by the process just described.

We now need to know the directions of zonal propagation in the prototype. As we have already noted, the eigenvalue problem (with $\partial \mathbf{U}/\partial t$ neglected) for perturbations $\mathbf{U}_1$ of the basic state (4.10) is equivalent to the eigenvalue problem (with $\partial \mathbf{U}_{1p}/\partial t$ retained) for perturbations $\mathbf{U}_{1p} = \mathbf{u}_{1p}\, e^{i(k\phi - \sigma_p t)}$ of a homogeneous non-magnetic fluid (Poincaré 1910). The non-dimensional eigenfrequencies $\lambda = -\sigma_p/\Omega$ in Poincaré's problem for a sphere are determined by

$$(n\lambda + 2k)\, P_n^k(\tfrac{1}{2}\lambda) = 2(n+k)\, P_{n-1}^k(\tfrac{1}{2}\lambda), \tag{4.11}$$

where n is any integer greater than or equal to $k+1$ and P_n^k denotes the associated Legendre function of degree n and order k (cf. Greenspan 1968, p. 64). The roots λ are real and lie between ± 2. Correspondingly, the Alfvén waves for the prototype are neutrally stable. As regards their speed, Malkus (1967) showed that

$$c_R = \begin{cases} b_0^2(k+2)\,(k-1)/2\mu_0\rho_0\,\Omega \quad \text{for} \quad n = k+1, & (4.12) \\ b_0^2\left[k(k+2)\left\{1 \pm \left\{\dfrac{(k+1)\,(k+3)}{2k+3}\right\}^{\frac{1}{2}}\right\}^{-1} - 2\right]\Big/ 2\mu_0\rho_0\,\Omega \quad \text{for} \quad n = k+2. & (4.13) \end{cases}$$

A few values of c_R for the lower-order Alfvén waves, determined from (3.3) and (4.11)–(4.13), are given in table 2. If n is large, k is fixed and both $n\theta$ and $n(\pi-\theta)$ are large, $0 < \theta < \pi$, then

$$\frac{P_{n-1}^k(\cos\theta)}{P_n^k(\cos\theta)} = \frac{(n+\frac{1}{2})\{\sin[(n-\frac{1}{2})\,\theta + l\pi] + (k^2-\frac{1}{4})\,(2n\sin\theta)^{-1}\cos[(n+\frac{1}{2})\,\theta + l\pi] + O(n^{-2})\}}{(n+k)\,\{\sin[(n+\frac{1}{2})\,\theta + l\pi] + (k^2-\frac{1}{4})\,(2n\sin\theta)^{-1}\cos[(n+\frac{3}{2})\,\theta + l\pi] + O(n^{-2})\}}, \tag{4.14}$$

where

$$l = \tfrac{1}{4}(2k+1)\,\pi \tag{4.15}$$

(Erdélyi *et al.* 1953, p. 147), and (3.3) and (4.11) yield

$$\lambda = 2\cos\theta, \quad c_R = b_0^2(k\sec\theta_0 - 2)/2\mu\rho_0\,\Omega, \tag{4.16}$$

where

$$\theta = (n+\tfrac{1}{2})^{-1}\left\{\nu\pi - n^{-1}\cot\left(\frac{2\nu\pi}{2n+1}\right)\left[\frac{k^2}{2} + \frac{3}{8} - k\sec\left(\frac{2\nu\pi}{2n+1}\right)\right] + O(n^{-2})\right\}, \tag{4.17}$$

$$\nu = \tfrac{1}{4}(4N - 2k + 1)\,\pi \tag{4.18}$$

and N is an arbitrary integer. When $n\theta$ or $n(\pi-\theta)$ is not large, a different asymptotic approximation to $P_{n-1}^k(\cos\theta)/P_n^k(\cos\theta)$ is appropriate (Erdélyi *et al.* 1953, p. 147), and instead of (4.17) and (4.18) we get, after some algebra,

$$n\theta = \lambda_s(1 - 1/2n) + O(n^{-2}), \quad n(\pi-\theta) = \mu_s(1 - 1/2n) + O(n^{-2}), \tag{4.19}$$

where λ_s and μ_s are the sth positive roots of the respective equations

$$\lambda_s J_k'(\lambda_s) = -kJ_k(\lambda_s), \quad \mu_s J_k'(\mu_s) = kJ_k(\mu_s). \tag{4.20}$$

$k \backslash n$	2	3	4	5
1	1	−0·1766	−0 8200	−1·1834
		1·5100	0·6122	−0·0682
			1·7080	1·0456
				1·8060
2		0·6666	−0·2320	−0·7634
			1·2320	0·4670
				1·4964
3			0·5	−0 2532
				1·0532
4				0·4

TABLE 1. Values of the eigenfrequencies $\lambda = -\sigma_p/\Omega$ for Poincaré's problem for a sphere determined from (4.11).

$k \backslash n$	2	3	4	5
1	0	−0·663	−2·220	−1·845
		−0·338	0·633	−15·66
			−0·415	−0·044
				−0·466
2		2	−9·621	−3·620
			0·623	3·283
				0·337
3			5	−12·85
				1·85
4				9

TABLE 2. Values of the phase speed $\Omega\mu_0\rho_0 c_R/b_0^2 = k/\lambda - 1$.

A few values of λ and of c_R for the lower-order modes, computed from (3.3) and (4.11), are listed in tables 1 and 2. For a given n and k, each table contains $m = n-k$ modes. Of these half, or more precisely,

$$[\tfrac{1}{2}m] = \begin{cases} \tfrac{1}{2}m, & m \text{ even}, \\ \tfrac{1}{2}(m+1), & m \text{ odd}, \end{cases} \tag{4.21}$$

have positive λ. These features extend to all k and n. To see this, we note that the zeros x_{nr}, $r = 1, 2, \ldots, n-k$, $|x_{nr}| < 1$, of $P_n^k(x)$ interlace with the zeros $x_{n-1\,r}$ of $P_{n-1}^k(x)$ and that the identity

$$\begin{aligned}(1-x^2)\,[P_n^k(x)]^2 \frac{d}{dx}\,[P_{n-1}^k(x)/P_n^k(x)] + (n-k)\,[P_n^k(x_{n-1\,r})]^2 \\ = 2n\int_{x_{n-1\,r}}^{x} P_{n-1}^k(x)\,P_n^k(x)\,dx, \quad x_{nr} < x < x_{n\,r+1},\end{aligned} \tag{4.22}$$

implies that the ratio $P_{n-1}^k(x)/(P_n^k(x)$ is monotonic decreasing between the successive zeros x_{nr} and $x_{n\,r+1}$. Also $P_n^k(-x) = (-1)^{k+n}P_n^k(x)$. Thus the graph of $y = P_{n-1}^k(x)/P_n^k(x)$ is like that of $-\tan x$ for even $k+n$ and like that of $\cot x$ for odd $k+n$. In either case, it intersects the line $y = (nx+k)/(n+k)$ just once between x_{nr} and x_{nr+1}. Thus there are $m-1 = n-k-1$ roots λ of (4.11) between $2x_{n1}$ and $2x_{nk-n}$, of which $[\frac{1}{2}m]-1$ are positive. At $x = \pm 1$,

$$P^k_{n-1}(x)/P^k_n(x) = \pm (n-k)/(n+k). \tag{4.23}$$

Consequently there is also a root of (4.11) for which $2x_{nn-k} < \lambda < 2$. The single root $\lambda = -2$ less than $2x_{n1}$ corresponds to solutions in which the reduced pressure is given by

$$\pi_{1p} = r^k(1+z)^{-k} \quad \text{or} \quad r^k(1-z)^{-k}. \tag{4.24}$$

However, these two solutions are singular at $z = \mp 1$ respectively and are not acceptable as modes. This leaves $m = n-k$ modes for given n and k, of which $[\frac{1}{2}m]$ have positive values of λ. The same holds for the Alfvén waves when $k \geq 2$. That is, for given n and k there are $m = n-k$ Alfvén modes, of which $[\frac{1}{2}m]$ travel eastwards for $k \geq 2$.

Provided that these modes deform continuously, their phase velocities c_R retain their signs. Malkus's (1967) conclusions for a sphere, with b_0 = constant and $\zeta_0 = \tau_0 = 0$, are thereby extended to a wider class of regions V and basic states. To the nearest integer, half of the admissible modes with a given zonal wavenumber $k \geq 2$ that derive from modes in the prototype with a given n travel eastwards and half travel westwards. The shelloidal modes (with $n = k+1$), which propagate eastwards, are seen to be atypical not only of the modes in a sphere but also of the family of admissible modes as a whole.

We turn next to the question of stability. The imaginary part of (4.1) yields

$$\operatorname{Im}(\sigma)\int_V \operatorname{Im} u\bar{v}\, d\tau = 0. \tag{4.25}$$

If the integral $\left(J = \int_V \operatorname{Im} u\bar{v}\, d\tau\right)$ were zero, ω_R would drop out of the real part of (4.1). The latter possibility is precluded by conditions (4.8) and (4.9), as we have already seen. When these conditions apply, J must be non-zero and $\operatorname{Im}\sigma$ must vanish. Accordingly, all the admissible Alfvén waves whose propagation speeds we have been discussing are neutrally stable. They are also, perforce, governed by equations which are hyperbolic in part, at least, of V.

For the purpose of establishing neutral stability, condition (4.9) is unduly restrictive. Relative to a new frame with angular velocity $\mathbf{\Omega}+\Omega'\hat{\mathbf{z}}$, where Ω' is an arbitrary constant very much less than Ω, the equations of motion have the same form as before and the zonal velocity in the basic state is $r(\zeta_0-\Omega')\,\hat{\boldsymbol{\phi}}$ (i.e. ζ_0 is changed to $\zeta_0-\Omega'$). Hence, instead of requiring (4.9) it suffices to require that

$$k^2b_0^2 > 4b_0^{-2}B_0^{*2}+rh_{0r}-\beta\tau_{0r}\sin\theta(k^2b_0^2+zh_{0z})A_3^{-1}, \tag{4.26}$$

where B_0^* denotes the magnitude of the maximum deviation of $b_0^2-\mu_0\rho_0\Omega\zeta_0$ from its mean value. Thus conditions (4.8) and (4.26) suffice to imply neutral stability and they also imply that the modal equations are hyperbolic at least somewhere in V.

Lastly, it should be noted that a preference for westward drift could still exist among the modes that have been excluded from the above discussion. A contrived example of this arises from imposing a uniform westward angular velocity ($\ll \Omega$) on the basic flow. If the angular velocity is large enough it leads to a bias towards westward drift; but it leads simultaneously to a breach of (4.9). Modes with wavenumber $k = 1$ have been excluded, and it might be noted that, of the ten lower-order modes with $k = 1$ and $n \leqslant 5$ for a sphere with $\zeta_0 = 0$, $b_0 =$ constant and $\tau_0 = 0$, eight travel westwards and only one travels eastwards (table 2). It is conceivable that some of the admissible modes do not deform continuously and that some modes are lost by virtue of σ becoming infinite. These matters require further study. The ambivalence in drift direction noted above applies to the extent that neither of these contingencies arises. There remains too the possibility that the unstable modes tend to drift westwards.

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